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NECESSARY AND SUFFICIENT CONDITION FOR LOCAL EXTREMA

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ABSTRACT

It is often of interest to find the global extrema of a given differentiable function. Unfortunately it is not easy to find the global extrema directly. Thus we set our sights a little lower and develop necessary and sufficient conditions for finding all local extrema.

KEYWORDS: Global Extrema, Optimization Theory, Algorithms

INTRODUCTION

Necessary and in some cases sufficient conditions for *X* to be a stationary point, were developed by Kuhn and Tucker [11]. Many of the algorithms for solving problem based on these conditions, and termination criteria concerned with recognizing when a stationary point has been reached are derived from them. Wilde and Beightler (1967) presented a development of the conditions based on constrained derivatives. The non-rigorous formulation given here uses the Lagrangian.

A Necessary Condition for Local Extrema

Given a function $f : I \to R$ with *I* open, it is desirable to develop sufficient conditions for local extrema to exist. Then the stationary points found can be examined to see if any of them are local extrema.

Theorem 1

If $f : I \to R$ is differentiable at x_1, in an open interval I, then f has a local extremum at $x_1 \in I \Rightarrow f'(x_1) = 0$.

Proof Suppose $x_1 \in I$ is a local minimum. By assumption $f'(x_1)$ exists, and

$$\lim_{\Delta x \to 0^+} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = f'(x_1). \quad (1)$$

But as x_1 is a local minimum, there exists a $\delta \in R^+$ such that for all Δx where $|\Delta x| \le \delta$

$$\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \ge 0$$

Hence that the first limit in (1) is non-negative. However, if

$$\Delta x < 0$$

As it is in the middle expression in (1), we have

$$\frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}\leq 0$$

Hence the second limit in (1) is non-positive. As $f'(x_1)$ must equal both of these limits, one negative, one nonnegative and one nonpositive, they must both be zero. Hence $f'(x_1)=0$

The proof when x_1 is a local maximum is similar

$$f'(x_7)=0$$

Definition 1

f has a point of inflection at x_1 if f has a stationary point at x_1 but f does not have a local extremum at x_1 and f' has a local extremum at x_1.This leads to a distinction between the stationary points of f.

Definition 2

f has critical point at x_1 if f has a stationary point at x_1 and x_1 is a local extremum for f but not for f'.Thus points of inflection are stationary points which are not critical points.

Sufficient Conditions for Local Extrema

In view of Theorem 1, it is desirable to develop sufficient conditions for local extrema to exist. Then the stationary points found can be examined to see if any of them are local extrema. To this end, consider the Taylor series expansion about a point $x_1 \in I$:

$$f(x_1+h)=f(x_1)+hf'(x_1)+\frac{h^2}{2}f''(\theta x_1+(1-\theta)(x_1+h)),$$

For some $\theta, 0<\theta<1,$ (2)

Where it is assumed that $(x_1+h)\in I$,

And *f has* first and second derivatives $f'(x_1)=0$ for all points in I. Suppose that f has a stationary point at x_1. Then by Theorem 1,

$$f'(x_1)=0.$$

Then (2) can be rearranged as:

$$f(x_1+h)-f(x_1)=\frac{h^2}{2}f''(\theta x_1+(1-\theta)(x_1+h)),\quad 0<\theta<1 \tag{3}$$

Assume

$$f''(x_1) > 0$$

Now if f'' is continuous on I, at all points sufficiently near x_1, f'' will have the same sign as it does at x_1, i.e. positive .Thus, for all h sufficiently small in magnitude,

$$\frac{h^2}{2} f''(\theta x_1 + (1-\theta)(x_1 + h)) > 0.$$

Using this result in (3), we obtain

$$f(x_1 + h) - f(x_1) > 0.$$

We conclude that if

$$f'(x_1) = 0.$$

And f'' is continuous in a neighbourhood of x_1, then

$$f''(x_1) > 0$$

is a sufficient condition for x_1 to be a local minimum.

It can be shown analogously that if

$$f'(x_1) = 0,$$

And f'' is continuous in a neighbourhood of x_1, then

$$f''(x_1) < 0$$

is a sufficient condition for x_1 to be a local maximum.

The preceding deductions cannot be used to come to any conclusions about the character of x_1 if

$$f'(x_1) = f''(x_1) = 0.$$

Indeed, it may be that

$$f^n(x_1) = 0, \qquad n = 1,2,...,k$$

For some integer k>2 the following theorem settles such cases.

Theorem 2

If $f^*(x_1) = 0,\ \mathrm{n} = 1,2,3.....\mathrm{k}$ (4)

and

$$f^{k+1}(x_1) \neq 0 \tag{5}$$

And $f^{(k+1)}$ is a continuous in a neighborhood of x_1, then f has a local extremum at x_1 if and only if (k+1) is even. If

$$f^{(k+1)}(x_1) > 0$$

x_1 is a local minimum. If

$$f^{(k+1)}(x_1) < 0$$

x_1 is a local maximum.

Proof: (Sufficient Condition)

We assume the hypothesis of Theorem 2. We wish to show that f has a local extremum at x_1.Taylor's Theorem about point x_1 yields

$$f(x_1 + h) = f(x_1) + hf'(x_1) + \frac{h^2}{2} f''(x_1) + \ldots \frac{h^{k+1}}{(k+1)!} f^{(k+1)}(\theta x_1 + (1-\theta)(x_1 + h)),$$

For some $\theta, 0 \leq \theta \leq 1$ Using (4) and rearranging, this becomes

$$f(x_1 + h) - f(x_1) = \frac{h^{k+1}}{(k+1)!} f^{(k+1)}(\theta x_1 + (1-\theta)(x_1 + h)),\ 0 \leq \theta \leq 1 \tag{6}$$

It is assumed that $f^{(k+1)}$ is continuous at x_1. This fact can be used to show that at all points sufficiently near x_1, $f^{(k+1)}$, will have the same sign as $f^{(k+1)}, x_1$. Hence if h is sufficiently small, $f^{(k+1)}(\theta x_1 + (1-\theta)(x_1 + h))$ will have the same sign as $f^{(k+1)}(x_1)$.

In view of this, on examining (6) we can see that odd k+1 and sufficiently small positive h, $f(x_1 + h) - f(x_1)$ has the opposite sign to $f^{(k+1)}(x_1)$. Hence, for odd k+1, x_1 is not a local extremum. However, if k+1 is assumed to be even, $f(x_1 + h) - f(x_1)$ has the same sign as $f^{(k+1)}(x_1)$, independently of the sign of h. If

$$f^{(k+1)}(x_1) > 0$$

Then

$$f(x_1 + h) - f(x_1) > 0$$

For all h sufficiently small in magnitude and thus x_1 is a local minimum. If

$$f^{(k+1)}(x_1) < 0$$

Then $f(x_1 + h) - f(x_1) < 0$

For all *h* sufficiently small in magnitude and thus x_1 is a local maximum

We assume (4) and (5) and that f has a local extremum at x_1.We wish to show that k+1 is even. Let us suppose for definiteness that f has a local minimum at x_1 i.e.

$$f(x_1+h)-f(x_1)>0$$

for all h sufficiently small in magnitude. Using (6) we obtain

$$\frac{h^{k+1}}{(k+1)!}f^{(k+1)}(\theta x_1+(1-\theta)(x_1+h))>0,\ 0<\theta<1 \tag{7}$$

I.e. the expression on the left-hand side of (7) is of constant sign, namely positive. However, from the arguments mounted earlier in the proof $f^{(k+1)}(\theta x_1+(1-\theta)(x_1+h))$ will have constant sign (it cannot be zero if (7) is to hold) for h sufficiently small in magnitude. Now when h is negative the expression in (7) can have constant sign only if k+1 is even.

A similar argument follows when f has a local maximum at x_1. This completes the proof.

CONCLUSIONS

We developed necessary and sufficient conditions for finding local extrema. Theorem 2 shows that f has a local extremum at x_1 if and only if (k+1) is even.

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